Unraveling the Mystery:
How is the Price of a Derivative Determined?

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Summary

This article provides an overview of common derivatives used in financial markets and their respective applications. The core mechanism of option pricing, the 'rule of no arbitrage,' is introduced, followed by an explanation of two methods for calculating option prices: the replicating portfolio method and the risk-neutral valuation method. Theoretical explanations and computational examples are provided for both approaches. The article concludes with an explanation of the practical advantages of the risk-neutral valuation method over the replicating portfolio method and what Amsshare can do with the knowledge provided in this article.

How can Amsshare support

The information provided in this article – from theoretical understanding of options to calculating option pricing methods – is well-known by Amsshare. Hence, Amsshare can support firms with projects within this area.

Chapter 1: Introduction to derivatives

A derivative is a financial instrument whose value depends on, or is derived from, the value of other more basic underlying variables. The underlying variables can be stocks, bonds, exchange rates, interest rates, etc. The fluctuation of the value of the derivatives is derived from the fluctuation of the value of the underlying. A derivative is also known as a contingent claim: the value of the derivative is contingent (dependent) on the value of the underlying.

Types of derivatives

The most commonly traded derivatives are the following three types:

- *Forwards/futures*: both the buyer (long) and seller (short) cannot make choices in the future. They are obliged to exercise the forward/future for the price they have agreed upon today.
- *Options*: the buyer (long) can choose in the future whether to exercise or not, while the seller (short) has no choice (has to fulfill the contract). There are three types: i) *European option*: this option can only be exercised at maturity ii) The *American option*: this option can be exercised any moment throughout the maturity of the option iii) *Exotic options*: this is an option with a more complex structure in terms of its payoff structure (depends on the type of exotic option, there are many).
- *Swaps*: This is a prearranged future agreement between two parties to exchange cash flows according to, for example, interest rates, currency rates, etc. For example, fixed vs. floating interest rate cash flows. Fixed cash flows are paid at every point in time and floating cash flows are received, or vice versa.

The primary focus of this article is on **European equity options**, but the principles discussed can be applied to other underlying assets. Our aim is to provide a clear understanding of pricing methods, and to achieve this, we utilize a straightforward framework with discrete timesteps and a binomial tree. The use of a binomial tree facilitates the comprehension of derivative pricing procedures. It is worth noting that derivatives are commonly priced in continuous time in practice, with the Black-Scholes model being a notable example.

Purposes for the use of derivatives

The use of derivatives in the financial world is an important aspect of financial institutions' balance sheets. As derivatives are specially structured with their payoff dependent depending on the type of underlying and the type of derivative contract (see *types of derivatives*) it can be used for different purposes:

- Hedging: Derivatives can be used to manage or hedge risks. Think about a diversified portfolio of an asset manager with exchange rate risk. The asset manager could use derivatives to hedge against fluctuations in exchange rates that could negatively impact their business. The same holds for owning a specific stock. Fluctuations in the stock can be hedged by using derivatives.

- Speculation: Derivatives can be used to speculate on the future movements of underlying assets, allowing traders to potentially profit from changes in market conditions.
- Arbitrage: Derivatives can be used for arbitrage, which involves exploiting price differences between two or more markets. More about arbitrage later in the article.
- *Financial Engineering*: using derivatives, you can construct portfolios whose payoffs match individual demands.

European Call - and Put Options

Widely traded equity options are call options and put options. It is important to understand these two options, such as their structure as it helps for the interpretation of options in general. Call and put options are both popular options.

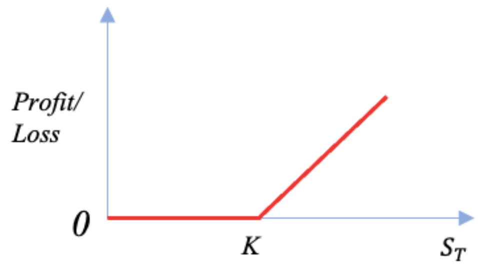
Call option

A call option gives the buyer the right, but not the obligation, to buy the underlying asset at a predetermined price (strike price) at maturity/expiration.

Long Call

The payoff of a long position in a call option is the difference between the market price (S) of the underlying asset and the strike price (K), but only if the market price is greater than the strike price. If it is lower than the strike price, then the payoff is zero. Hence, the buyer of the call option makes a profit if the price of the underlying asset increases.

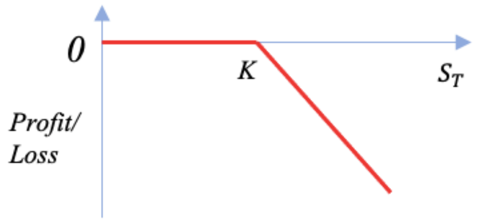
Figure 1: $Call\ payoff = \max(S-K,0) - premium\ per\ share$



Short Call

The payoff of a short position in a call option is the opposite of the long call. It is used to hedge against a decrease in the underlying (stock) price. If the stock price falls below the strike price, the option is worthless. Then the writer receives a premium without losing money. However, the seller of the call option has unlimited risk exposure as the option may keep decreasing. To sum up, the seller/writer of the call option makes a profit if the price of the underlying asset decreases.

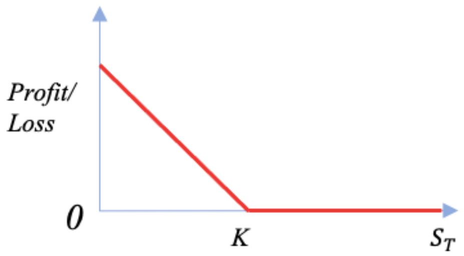
Figure 2: $Call\ payoff = \max(0, S - K) + premium\ per\ share$



Put option
A put option gives the buyer the right, but not the obligation, to sell the underlying asset at a predetermined price (strike price) at maturity/expiration.

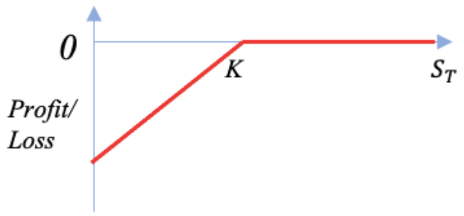
Long Put
The payoff of a long position in a put option is the difference between the strike price and the market price of the underlying asset, but only if the market price is lower than the strike price. If the market price is greater than the strike price the payoff is zero. Thus, the buyer of a put option profits when the market price of the underlying asset decreases.

Figure 3: $Put\ payoff = \max(K - S, 0) - premium\ per\ share$



Short Put
A short position in a put option is also called a naked put. It refers to when a trader opens an options trade by selling/writing a put option. The seller of the put, thus the short-positioned investor, exchanges an obligation for a premium. When the buyer, thus the long-positioned investor of the put exercises the option, the seller needs to repurchase the option. Hence, the seller receives a premium from the buyer in exchange for the obligation to purchase the underlying asset at the strike price if the buyer exercises their option. So, the idea is to generate profit from a stock price increase by collecting the premium from the short put sale.

Figure 4: $Put\ payoff = \max(0, K - S) + premium\ per\ share$



Chapter 2: Pricing derivatives by means of no arbitrage

The core mechanism of pricing financial instruments is 'pricing by no arbitrage'. This means that prices in an efficient market must satisfy the condition that no arbitrage opportunities exist. If this is the case, it cannot be possible that one earns money without (1) investing money today and/or (2) without the risk of losing money in the future. A more practical way of explaining the principle of arbitrage is that one can take advantage of the price difference of the same product in two different markets. It is a combination of matching deals that capitalize upon the imbalance where the profit is the difference between two market prices. If such an arbitrage opportunity exists, one can buy low, sell high and keep the difference. Then there is a 'free lunch' or 'free profit'. For pricing financial instruments it is assumed that the principle of 'no arbitrage' holds and, thus, that markets are efficient.

Example of exploiting an arbitrage opportunity

Let's assume that the stock 'ASML Holding NV' is listed in Amsterdam and in London. The price in Amsterdam is €600 and the price in London is €605 (converted to euros). Exploiters of the arbitrage opportunity (also called: arbitrageurs) will buy ASML in Amsterdam and sell (or shorten) in London. Arbitrageurs buy low, sell high and keep the difference. In this example arbitrageurs can achieve a 'free profit' of €5. Arbitrageurs will keep doing this until the markets are, by means of the principle of supply and demand, in equilibrium again.

Two ways of pricing derivatives

The two ways of pricing derivatives in this article are (1) replicating portfolio and (2) risk-neutral pricing.

Pricing methodology 1: Replicating portfolio

Since prices of financial instruments must satisfy the condition of 'no arbitrage', the price of a derivative must be equal to the price of a portfolio that perfectly replicates the payoff of the derivative. This means that the future value of the derivative and the replicating portfolio in each state of the world must be equal. If this does not hold, arbitrage opportunities exist and markets are inefficient.

For simplicity, the replicating portfolio consists of two assets: delta amount of stocks and B amount of money in the risk-free rate. If delta amount of stocks is negative or positive, it means that one goes short or long in the stocks, respectively. If B amount of money in the risk-free rate is negative or positive, it means that one takes a loan or invests money in the bank, respectively. The combination of these two assets need to replicate the payoff of the derivative that one wants to price. More specifically, this replicating portfolio needs to have the same price as the derivative in each state of the world. This means that the prices are equal in the case that stocks decrease in value (down-state), but also when stocks increase in value (up-state). In this article a call option is priced. As a result of this principle, the following formulas for the call option need to hold:

(1) Up-state: $\Delta * S_0 u + B * (1 + r) = C_1^u$

(2) Down-state: $\Delta * S_0 d + B * (1 + r) = C_1^d$

Where Δ = amount of stocks, S_0 = stock price at t = 0, u = up-state factor, d = down-state factor, B = amount of money invested in the bank, r = risk-free rate, C_1^u = call price up-state at t = 1 and C_1^d = call price down-state at t =1.

The Δ and B are the only unknowns in the formulas. The other quantities can be obtained in the market. Hence, by rewriting the formula above, the formulas for Δ and B can be retrieved:

(3) $\Delta = \frac{C_1^u - C_1^d}{S_0 u - S_0 d}$

(4) $B = \frac{C_1^u - \Delta * S_0 u}{1+r}$

By using the formulas (1) to (4), the call option prices can be calculated for the up-state and down-state in the world.

Example replicating portfolio – theory
In this example, a call option will be priced by means of the replicating portfolio. The call option has a strike price of €105. The current stock price (t = 0) is €100. The up-factor is 1.2 and the down-factor is 0.8. The risk-free rate is 5%.

Firstly, a binomial tree can be created with the information that is available. This is shown in Figure 5. Figure 5 shows the stock price simulation and the payoff of the call option at maturity (t = 2). To obtain the call option prices for t = 1, the formulas of delta (3) and B (4) need to be applied. After that, the formulas for the up-state (1) and down-state (2) can be used to obtain the call prices at t = 1 in the up-state and down-state, respectively. With this mechanism, one can calculate the option price for t = 0 by calculating the option values from right to left through the binomial tree until the value for t = 0 is reached. The state of t = 0 is not an up-state nor a down-state since it is the current state, hence the following formula for the call option can be used: $\Delta * S_0 + B * (1 + r) = C_0$

Example replicating portfolio – calculations
Since the strike price of the call option is €105, the payoffs at maturity are 39, 0 and 0. With this information, the call option prices at t = 1 can be calculated for the up-node and down-node.

The calculation for the call option price at t =1 in the up-node is (C_1^u):

1. Calculation of delta: $\Delta = \frac{39-0}{144-96} = 0.8125$
2. Calculation of B: $B = \frac{39-(0.8125*144)}{1.05} = -74.2857$
3. Call option price at t = 1 (up-node): $-74.2857 + 0.8125 * 120 = 23.21$

The same calculation can be performed for the call option price at t = 1 in the down-node and for the price at t = 0:

The calculation for the call option price at t = 1 in the down-node is (C_1^d):

1. Calculation of delta: $\Delta = \frac{0-0}{96-64} = 0$
2. Calculation of B: $B = \frac{0-(0*96)}{1.05} = 0$
3. Call option price at t =1 (down-node): $0 + 0 * 80 = 0$

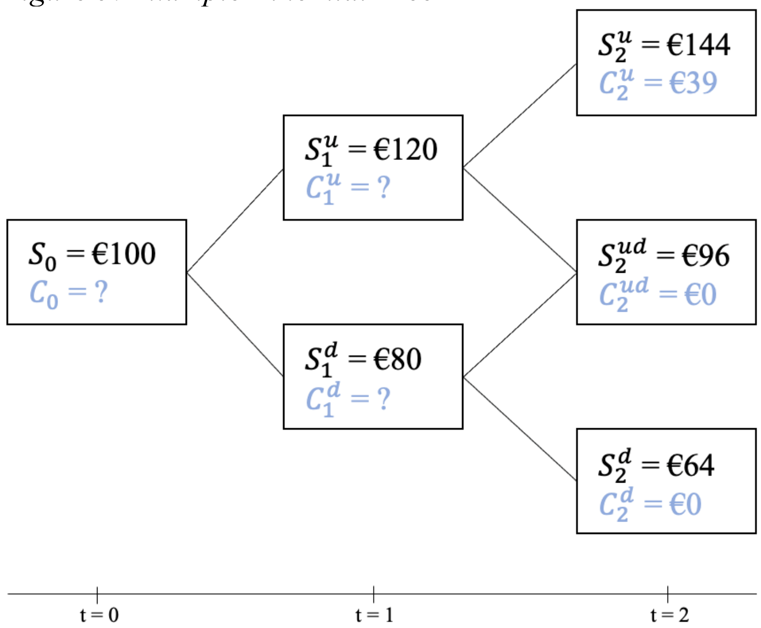
The calculation for the call option price at t = 0 (C_0):

1. Calculation of delta: $\Delta = \frac{23.21-0}{120-80} = 0.5803$

2. Calculation of B: $B = \frac{23.21-(0.5803*120)}{1.05} = -44.2152$
3. Call option price at t =1 (down-node): $-44.2152 + 0.5803 * 100 = 13.82$

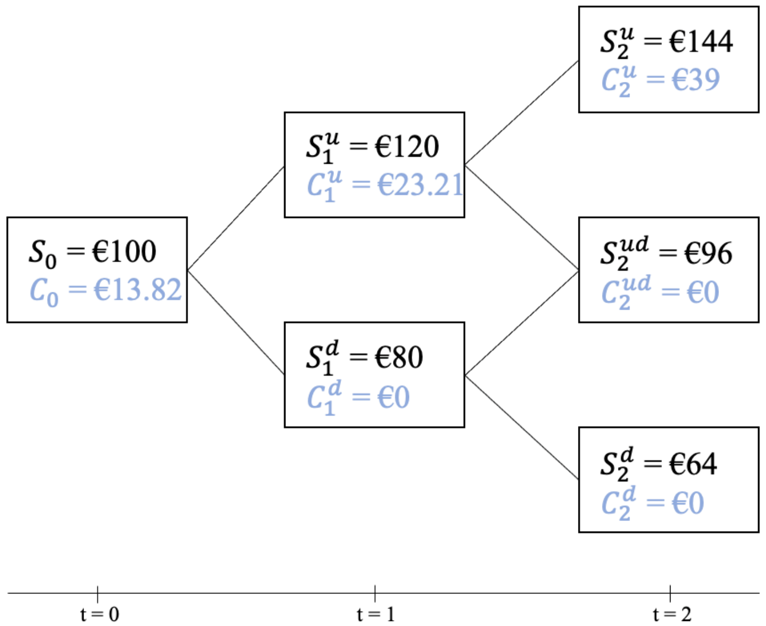
So, by calculating the tree from right to left the call option price can be calculated for t = 0. Figure 6 shows the results. The call option price is €13.82. This calculation shows that one can calculate the price of an option by using the replicating portfolio (consisting of delta amount of stocks and B amount of money in the risk-free rate).

Figure 5: Example Binomial Tree



Start stock price (S0) =€100, Strike price (K) = 105, risk-free rate (rf) = 0.05, up-state = 1.2 and down-state = 0.8
Note: S_2^{ud} and C_2^{ud} could also be S_2^{du} and C_2^{du}, because both paths end up in the same node

Figure 6: Solution Binomial Tree

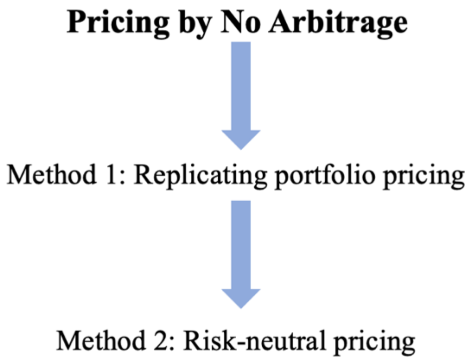


Start stock price (S0) =€100, Strike price (K) = 105, risk-free rate (rf) = 0.05, up-state = 1.2 and down-state = 0.8
Note: S_2^{ud} and C_2^{ud} could also be S_2^{du} and C_2^{du}, because both paths end up in the same node

Pricing methodology 2: Risk-Neutral Pricing

Risk-neutral pricing is a simplified method of calculating option prices based on the replicating portfolio method. The basis of the replicating portfolio method is pricing by no-arbitrage, which therefore also holds for risk-neutral pricing. The following figure illustrates the relation between these three: overall pricing by no arbitrage is the theory that is the basis for both pricing methods.

Figure 3: Relation of the two pricing methods



The idea of this pricing method is that we calculate the risk-neutral probabilities (named **q**) of the option in every point in time of the option. Note that it is not called risk-neutral pricing because of a risk-neutral trading preference but it refers to pricing using the risk-free rate as discounting factor. The risk-neutral pricing method is just a rewritten equation of the replicating portfolio method and thus is based on the no arbitrage theory. For this theory to hold investors are non-satiated; investors are greedy and want to make as much profit by arbitrage as possible.

The replicating portfolio pricing formula: $C_0 = \Delta * S_0 + B$ can be rewritten to the following form: $C_0 = \frac{1}{1+r}\left(\frac{1+r-d}{u-d}C_1^u + \frac{u-(1+r)}{u-d}C_1^d\right)$ (If you are interested in the derivation of this formula, we are happy to share it with you. Just send us a message on our website or send an email). Here $\frac{1+r-d}{u-d}$ = ***q*** and $\frac{u-(1+r)}{u-d}$ is (***1 - q***).

The derivative price has both an up-state and down-state. We calculate these prices by multiplying the up-state with the calculated risk-neutral probability ***q*** and the down-state with (***1 - q***). In the price path of an option the risk-neutral probabilities are equal in every node. Note that these probabilities are artificial probabilities and have nothing to do with real probabilities in the equity trading market.

As explained before, to calculate the option price at t=0 we have to start at the maturity/expiration of the option and calculate the payoffs at this end of the tree.

The steps of this pricing method are:

1. Calculate risk-neutral probability q
2. Calculate option price at T (equal to the payoff at T)
3. Calculate option price at step t=1; one future step after t=0 (use risk-neutral pricing formula and discount it with the risk-free rate)
4. Calculate the option price at t=0

As an example, we use the binomial tree again to clarify the method (see solution in *figure 6*):

1. Q = 0.625 and (1-q) = 0.375
2. $C_2^u = \max(S_T - K, 0) = €144 - €105 = €39$
   $C_2^{ud} = \max(S_T - K, 0) = €96 - €105 = €0$
   $C_2^d = \max(S_T - K, 0) = €64 - €105 = €0$
3. $C_1^u = \frac{1}{1.05}(0.625 * 39 + 0.375 * 0)$ = €23.21
   $C_1^d = \frac{1}{1.05}(0.625 * 0 + 0.375 * 0)$ = €0
4. $C_0 = \frac{1}{1.05}(0.625 * 23.21 + 0.375 * 0)$ = €13.82

Advantage of risk-neutral pricing method over the replicating portfolio method

The calculation above shows how to calculate the option price at every node in the tree. However, the risk-neutral pricing measure comes with a significant advantage compared to the replicating portfolio method. This advantage is that we can calculate the option price at t=0 by only using the payoffs at maturity, use risk-neutral probabilities, and then discount those with the risk-free rate. This means that you can calculate the option value in one formula, instead of calculating the replicating portfolio for each state and node in the tree.

How does that look like in this example:

- ➔ The payoffs at maturity are €39 (uu), €0 (ud, du) and €0 (dd), respectively.
- ➔ $C_0 = \frac{1}{1.05^2}((0.625 * 0.625 * 39) + (0.625 * 0.375 * 0) + (0.375 * 0.625 * 0) + (0.375 * 0.375 * 0)) = €13.82$

The calculation above shows that you can calculate the option value for t=0 in one formula. This is not possible when you use the replicating portfolio. Hence, for larger and more complex binomial trees it is more practical to use the risk-neutral valuation method.